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Total Positivity, Interpolation by Splines, and Green's Functions of Differential Operators*†

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A polynomial spline of order m (degree $m-1$) with knots $\{\xi_i\}_{i=1}^{r}$ $(0 < \xi_1 < \xi_2 < \cdots < \xi_r < 1)$ has the form

$$S(x) = \sum_{i=0}^{m-1} a_i x^i + \sum_{i=1}^{r} c_i (x - \xi_i)_+^{m-1} \tag{0.1}$$

where, as usual, $x_+^{\,k} = x^k$ for $x \geqslant 0$, $x_+^{\,k} = 0$ for $x < 0$, and a_i, c_i are real constants. Fundamental to the study of interpolation and approximation by splines on the interval [0, 1] (see Section 2) is the kernel $K(z, w)$ defined on $Z \times W$, where Z and W are the specific *ordered* sets (consisting of a set of integers and points of an open interval)

$$Z = \{x, 0, 1,..., m-1; x \in (0, 1)\}$$

and

$$W = \{0, 1, 2,..., m-1, \xi; \xi \in (0, 1)\}.$$

$K(z, w)$ is defined as follows:

$$\begin{aligned}
&K(x, i) = u_i(x) = x^i,\\
&K(x, \xi) = (x - \xi)_+^{m-1},\\
&K(j, \xi) = D_x{}^j \phi(x, \xi)\,|_{x=1} \quad \left(\phi(x, \xi) = (x - \xi)_+^{m-1}, D = \frac{d}{dx}, D^j = D^{j-1} D\right),\\
&K(j, i) = D^j u_i(x)\,|_{x=1}\,.
\end{aligned} \tag{0.2}$$

* Part of the contents of this paper is reviewed in Karlin [4].

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(Note that in the domain Z the integers are arranged to occur after the x values, while in W they are placed prior to the ξ values.)[1]

The kernel $K(z, w)$ has remarkable total positivity properties, elaborated in Sections 1 and 2 and decisive in the ascertainment of complete criteria for interpolating arbitrary data at given points by splines with prescribed knots. These interpolation results, of independent interest, will serve in determining optimal quadrature formulas under a variety of circumstances.

Theorem 1 of Section 1 will also prove indispensable to our investigations (Karlin and Karon [6]) of the Birkhoff interpolation problem (see Schoenberg [10], Ferguson [2], Atkinson and Sharma [1], and Lorentz and Zeller [9] for recent contributions concerning this problem).

Furthermore, the total positivity structure of $K(z, w)$ plays a key role in analyzing best L^2 approximation to functions by splines with variable knots (see Karlin [5]).

For other purposes it is important to construct the analog of the kernel (0.2) associated with certain generalized differential operators. To this end, let $\{w_i(x)\}_1{}^m$ be positive and of class C^m on [0, 1]. Consider the m-th order differential operator

$$L_m v = D_m D_{m-1} \cdots D_1 v \tag{0.3}$$

composed from the first-order differential operators

$$(D_i v)(x) = \frac{d}{dx} \frac{1}{w_i(x)} v(x), \qquad i = 1, 2, \ldots, m,$$

acting on $v \in C^m[0, 1]$. The solutions of $L_m v = 0$ analogous to the powers $\{x^i\}_0^{m-1}$ are $\{u_i(x)\}_0^{m-1}$, where

$$u_0(x) = w_1(x),$$

$$u_i(x) = w_1(x) \int_0^x w_2(t_1) \int_0^{t_1} w_3(t_2) \cdots \int_0^{t_{i-1}} w_{i+1}(t_i)\, dt_i\, dt_{i-1} \cdots dt_1, \qquad i = 1, 2, \ldots, m-1 \tag{0.4}$$

(see Karlin, [3, p. 27, and Chapter 6]). We let $\varphi_m(x; \xi)$ be the fundamental solution of $L_m v = 0$ whose explicit representation is

$$\phi_m(x; \xi) = \begin{cases} 0, & x < \xi, \\ w_1(x) \int_\xi^x w_2(t_1) \int_\xi^{t_1} w_3(t_2) \cdots \int_\xi^{t_{m-2}} w_m(t_{m-1})\, dt_{m-1} \cdots dt_1, & x \geqslant \xi. \end{cases} \tag{0.5}$$

[1] Observe the difference in notation from Karlin [4]; Z and W are interchanged to conform with the notation employed in the book by Karlin [3].

A "generalized spline" associated with the differential operator L_m and exhibiting knots $\{\xi_i\}_{i=1}^r$ takes the form

$$S(x) = \sum_{i=0}^{m-1} a_i u_i(x) + \sum_{i=1}^{r} c_i \phi_m(x; \xi_i).^2 \tag{0.6}$$

The extended form of the kernel (0.2) associated with the differential operator L_m defined on $Z \times W$ becomes

$$\begin{aligned} K(x, i) &= u_i(x), \\ K(x, \xi) &= \phi_m(x; \xi), \\ K(j, \xi) &= D_x{}^j \phi_m(x; \xi) \mid_{x=1}, \\ K(j, i) &= D^j u_i(x) \mid_{x=1}. \end{aligned} \tag{0.7}$$

(Here $D^j = D_j D_{j-1} \cdots D_1$, $D^0 = I =$ identity operator.) We recover (0.2) by the specification $w_1(x) \equiv 1$, $w_i(x) \equiv i - 1$, $0 \leqslant x \leqslant 1$. All the results elaborated for the kernel (0.2) persist for the generalized kernel (0.7), mutatis mutandis. To ease the exposition we will concentrate mainly on the kernel (0.2) and accordingly deal with the polynomial splines $S(x)$ of (0.1).

The total positivity nature of the kernel (0.7) is vital for deducing fine properties of the Green's function for the differential operator L_m coupled with boundary conditions at the end points 0 and 1 of the form (0.8) below. The determination of the Green's function with the desired properties is developed in Section 4.

Consider the homogeneous boundary conditions

$$\begin{aligned} \beta_0 &: \sum_{\mu=0}^{m-1} A_{\nu\mu} D^\mu S(0) = 0, \qquad \nu = 1, 2, \ldots, p, \\ \beta_1 &: \sum_{\mu=0}^{m-1} B_{\lambda\mu} D^\mu S(1) = 0, \qquad \lambda = 1, 2, \ldots, q. \end{aligned} \tag{0.8}$$

A spline of the kind (0.1) fulfilling the boundary conditions $\beta_0 \cap \beta_1$ is said to be of class $\mathscr{S}_{m,r}(\beta_0 \cap \beta_1)$.

The following requirements are assumed to prevail unless stated otherwise.

POSTULATE I. (i) $p + q \leqslant m$.

(ii) *The* $p \times m$ *matrix* $\tilde{A} = \| A_{\nu\mu}(-1)^\mu \|$ *is sign consistent of order* p (SC_p) *and has rank* p (*a matrix* U *is said to be* SC_p *if all* $p \times p$ *nonzero subdeterminants of* U *have the same sign*).

(iii) *The* $q \times m$ *matrix* $B = \| B_{\lambda\mu} \|$ *is* SC_q *and of rank* q.

[2] In Karlin [3] we refer to (0.6) as a Tchebycheffian spline.

Several concrete illustrations of boundary conditions fulfilling Postulate I are indicated in the companion paper to this (see also section 3).

In that section the following general interpolation problem is treated. Let $\{x_j\}_1^\gamma$ satisfy $0 < x_1 < x_2 < \cdots < x_\gamma < 1$ and let $m + r = \gamma + p + q$. When is it possible to interpolate arbitrarily preassigned values $\{y_i\}_1^\gamma$ at the points $\{x_j\}_1^\gamma$ by a spline of class $\mathscr{S}_{m,r}(\beta_0 \cap \beta_1)$? The exact criterion when such an interpolation is possible is indicated in Theorem 2.

We now fix some notation. If $A = \| A_{ij} \|$ then

$$A\begin{pmatrix} i_1, i_2, \ldots, i_p \\ j_1, j_2, \ldots, j_p \end{pmatrix}$$

denotes the minor of A composed of rows and columns of indices

$$i_1 < i_2 < \cdots < i_p \qquad \text{and} \qquad j_1 < j_2 < \cdots < j_p ,$$

respectively. For $z_1 < z_2 < \cdots < z_p$; $w_1 < w_2 < \cdots < w_p$,

$$K\begin{pmatrix} z_1, z_2, \ldots, z_p \\ w_1, w_2, \ldots, w_p \end{pmatrix}$$

will denote the corresponding Fredholm minor based on the kernel $K(z, w)$.

We will exploit frequently the Sylvester determinant identity which is quoted here for easy reference:

Let A be a fixed $n \times n$ matrix. Let $1 \leqslant \nu_1 < \nu_2 < \cdots < \nu_p \leqslant n$ and $1 \leqslant \mu_1 < \mu_2 < \cdots < \mu_p \leqslant n$ be two p-tuples of indices to be held fixed.

For every index i $(1 \leqslant i \leqslant n)$ not contained in the set $\nu = (\nu_1, \nu_2, \ldots, \nu_p)$ and every index j $(1 \leqslant j \leqslant n)$ not contained in the set $\mu = (\mu_1, \mu_2, \ldots, \mu_p)$, we form

$$b_{ij} = A\begin{pmatrix} k_1, k_2, \ldots, k_{p+1} \\ l_1, l_2, \ldots, l_{p+1} \end{pmatrix}$$

where $(k_1, k_2, \ldots, k_{p+1})$ is the set of indices $(i, \nu_1, \nu_2, \ldots, \nu_p)$ arranged in increasing order, and $(l_1, l_2, \ldots, l_{p+1})$ is the set of indices $(j, \mu_1, \mu_2, \ldots, \mu_p)$ arranged in increasing order. Then for $i_1 < i_2 < \cdots < i_q$, with each $i_m \notin \nu$, and for $j_1 < j_2 < \cdots < j_q$, with each $j_m \notin \mu$, where $q \leqslant n - p$, we have

$$B\begin{pmatrix} i_1, i_2, \ldots, i_q \\ j_1, j_2, \ldots, j_q \end{pmatrix} = \left[A\begin{pmatrix} \nu_1, \nu_2, \ldots, \nu_p \\ \mu_1, \mu_2, \ldots, \mu_p \end{pmatrix}\right]^{q-1} A\begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_{q+p} \\ \beta_1, \beta_2, \ldots, \beta_{q+p} \end{pmatrix},$$

where

$$(\alpha_1, \alpha_2, \ldots, \alpha_{q+p}) = (i_1, i_2, \ldots, i_q, \nu_1, \nu_2, \ldots, \nu_p),$$
$$(\beta_1, \beta_2, \ldots, \beta_{q+p}) = (j_1, j_2, \ldots, j_q, \mu_1, \mu_2, \ldots, \mu_p)$$

are each arranged in natural order.

We will use the above identity mostly for the case $q = 2$.

We conclude this introduction with a brief review of the organization of the paper.

Sections 1 and 2 are devoted, respectively, to the formulation and demonstration of the precise total positivity character of the kernels (0.2) and (0.7) Section 3 reports complete results concerning unique interpolation by splines satisfying rather general boundary conditions. Our study unifies, extends, and refines much of the previous work on this topic. Whereas most related developments focus on interpolating given data exclusively at the knots for special classes of boundary constraints, we have described general criteria on the knots, interpolatory points, and prescription of the boundary conditions for unique interpolation by splines. More specifically, Theorem 1 makes it possible to determine at exactly which points the interpolation problem is "poised." Interpretation of this result for certain physical systems is indicated in Chapter 10, Section 9 of Karlin [3].

The theorems of Section 4 describe the fine total positivity structure of a wide class of Green's functions associated with certain n-th order differential operators and related boundary conditions. Again, Theorem 1 is the key to this development.

1. Total Positivity Properties of the Kernel (0.2)

The principal result of this paper, bearing numerous consequences, is the following

THEOREM 1. (i) *The kernel* $K(z, w)$ *defined in* (0.2) *is totally positive* (TP): *For any sets* $\{x_t, j_v\}$ *and* $\{i_v, \xi_t\}$ *satisfying*

$$\begin{aligned} &0 < x_1 < x_2 < \cdots < x_\lambda < 1, \qquad && 0 \leqslant j_1 < j_2 < \cdots < j_\rho \leqslant m-1, \\ &0 \leqslant i_1 < i_2 < \cdots < i_\sigma \leqslant m-1, \qquad && 0 < \xi_1 < \xi_2 < \cdots < \xi_\tau < 1, \end{aligned} \tag{1.1}$$

and $\lambda + \rho = \sigma + \tau$, *we have*

$$K\begin{pmatrix} x_1, x_2, \ldots, x_\lambda, j_1, j_2, \ldots, j_\rho \\ i_1, i_2, \ldots, i_\sigma, \xi_1, \xi_2, \ldots, \xi_\tau \end{pmatrix} \geqslant 0. \tag{1.2}$$

(ii) *Strict inequality holds in* (1.2) *if and only if the indices and variables obey the following constraints;*

(a) *When* $\sigma \geqslant \lambda$,

$$\begin{aligned} &j_\mu \leqslant i_{\lambda+\mu}, \qquad && \mu = 1, 2, \ldots, \sigma - \lambda, \\ &x_\nu < \xi_{m-\sigma+\nu}, \qquad && \nu = 1, 2, \ldots, \lambda; \end{aligned} \tag{1.3}$$

(b) *When* $\sigma < \lambda$,

$$\begin{aligned} x_\nu &< \xi_{m-\sigma+\nu}, & \nu &= 1, 2, \ldots, \lambda, \\ \xi_\mu &< x_{\sigma+\mu}, & \mu &= 1, 2, \ldots, \lambda - \sigma. \end{aligned} \tag{1.4}$$

The conditions are to apply only when the subscripts are meaningful. Notice that $\sigma, \rho \leqslant m$.

Remark 1. Conditions (1.3) and (1.4) have a simple visual interpretation. In both cases, if there is a ξ_ν, m units to the right of an x_μ in the display in (1.2), then x_μ must be less than ξ_ν. If there is a j_μ above an i_ν (case a), then j_μ must not exceed i_ν; if there is an x_μ above a ξ_ν (case b), then x_μ must be greater than ξ_ν.

Remark 2. Theorem 1 extends to the case where coincident ξ_ν's are permitted (i.e., knots of higher multiplicity) and coincident x_ν values occur (i.e., the prescription of values of the function and some of its higher derivatives) with the usual convention for evaluating the determinant (1.2) in these circumstances (see Karlin, [3, pp. 47–48). A slight modification in the conditions (1.3) and (1.4) must be made when m coincidences occur in the collections $\{x_1, x_2, \ldots, x_\lambda\}$ and/or $\{\xi_1, \xi_2, \ldots, \xi_\tau\}$. The precise statement is as follows.

THEOREM 1′. *Instead of the stipulation* (1.1) *suppose, more generally,*

$$\begin{aligned} &0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_\lambda < 1, && 0 \leqslant j_1 < j_2 < \cdots < j_\rho \leqslant m - 1, \\ &0 \leqslant i_1 < i_2 < \cdots < i_\sigma \leqslant m - 1, && 0 < \xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_\tau < 1, \end{aligned} \tag{1.5}$$

restricted so that $\lambda + \rho = \sigma + \tau$ *and*

(α) *No more than m consecutive x's or ξ's coincide*;

(β) *At most* $m + 1$ *of the x's and ξ's together are equal to a given value.*

Then

(i) *The kernel* $K(z, w)$ *of* (0.2) *is totally positive, i.e.,* (1.2) *holds subject to* (α) *and* (β). (*When m consecutive x's* (ξ's) *agree the* $(m - 1)$*-th derivative in* (1.2) *is taken as a right* (*left*) *derivative.*)

(ii) *Strict inequality occurs in* (1.2) *if and only if, when* $\sigma > \lambda$, (1.3) *prevails and, when* $\sigma < \lambda$, (1.4) *holds with the two added exceptions*:

If $\tau \geqslant m$ *and* $\xi_{\nu+1} = \xi_{\nu+2} = \cdots = \xi_{\nu+m}$ *for some* ν *with* $\nu + m \leqslant \tau$, (1.2) *also holds with a strict sign if* $\xi_{\nu+1} = x_{\sigma+\nu}$. *If* $\lambda \geqslant m$ *and* $x_{\mu+1} = x_{\mu+2} = \cdots = x_{\mu+m}$ *for some* μ *with* $\mu + m \leqslant \lambda$, (1.2) *also holds with a strict sign if*

$$x_{\mu+m} = \xi_{\mu+m-\sigma}.$$

Remark 3. The assertions of Theorems 1 and 1′ apply without alteration to the generalized kernel (0.7) .

The last statement of Theorem 1′ concerning the exceptional circumstances of strict inequality in (1.2) with maximal sets of coincidences should be also appended to Theorem 1.1 of Karlin [3, Chapter 10].

Some special cases of Theorem 1 are worth highlighting.

EXAMPLE I. Suppose $\lambda = \tau = 0$ and, therefore, $\rho = \sigma$. Then the requirement for strict positivity in (1.2) reduces to

$$j_\mu \leqslant i_\mu, \qquad \mu = 1, 2,..., \sigma.$$

EXAMPLE II. Consider the special case where $\lambda = \tau$, $\rho = \sigma > 0$, and $\xi_i = x_i$ for all i. Then (1.2) always holds strictly when $\sigma < \lambda$. When $m > \sigma \geqslant \lambda$ the condition simply becomes

$$j_\mu \leqslant i_{\lambda+\mu}, \qquad \mu = 1, 2,..., \sigma - \lambda.$$

2. PROOF OF THEOREM 1

The proof of Theorem 1 is delicate, relying on the precise total positivity properties of a restriction of the kernel $K(z, w)$ already established in Karlin, [3, Chapter 10, Section 2], on several exploitations of the Sylvester determinant identity, and on a variety of double inductions (forward and backward).

The proof of part (i) also employs a standard smoothing argument. The proof of (ii) is accomplished by a consideration of cases, each case being basic for the others.

LEMMA 1. *Part* (i) *of Theorem* 1.

Remark. The proof to be given adapts the arguments of the proof of Theorem 2.1 in Karlin, Ref [3, Chapter 10].

Proof.

Define for $\epsilon > 0$, $G_\epsilon(x, y) = \dfrac{2}{\sqrt{2\pi}\,\epsilon} \exp\left[-\dfrac{1}{2\epsilon^2}(x - y)^2\right]$, $0 \leqslant x, y < \infty$ and set $z = \{x, (0 < x < 1)\ 0, 1,...\ m - 1\}$. Next define

$$H_\epsilon(z, y) = \begin{cases} G_\epsilon(x, y) & 0 < z = x < 1, \quad 0 \leqslant y < \infty, \\ D_u{}^\nu G_\epsilon(u, y)\,|_{u=1}, & 0 \leqslant y < \infty, \quad z = \nu, \quad \text{an integer.} \end{cases}$$

$G_\epsilon(x, y)$ is extended totally positive (ETP) (see Karlin, [3, Chapter 3, p. 103]); that is,

$$G_\epsilon{}^* \begin{pmatrix} x_1, x_2, ..., x_p \\ y_1, y_2, ..., y_p \end{pmatrix} > 0$$

for all choices of x_i and y_j in $(0, \infty)$ arranged in increasing order. Set

$$F_\epsilon(z, w) = \int_0^\infty H_\epsilon(z, y)\, K(y, w)\, dy$$

where $w \in W$, $z \in Z$. From the basic composition formula [3, Chapter 1, page 17],

$$\begin{aligned} &F_\epsilon\begin{pmatrix} \hat{x}_1, \dots, \hat{x}_p, 0, 1, \dots, s-1 \\ q, q+1, \dots, m-1, \hat{\xi}_1, \hat{\xi}_2, \dots, \hat{\xi}_r \end{pmatrix} \\ &\quad = \int \cdots \int_{0<y_1<\cdots<y_{p+s}<\infty} H_\epsilon\begin{pmatrix} \hat{x}_1, \dots, \hat{x}_p, 0, \dots, s-1 \\ y_1, \dots \quad , y_{p+s} \end{pmatrix} \\ &\quad\quad \times K\begin{pmatrix} y_1, \dots \quad , y_{p+s} \\ q, \dots, m-1, \hat{\xi}_1, \dots, \hat{\xi}_r \end{pmatrix} dy_1 \cdots dy_{p+s} \end{aligned} \tag{2.1}$$

where $r = p + s - (m - q)$. In the notation of Karlin, [3, Chapter 1], the first determinant under the integral sign is

$$G_\epsilon^*\begin{pmatrix} \hat{x}_1, \dots, \hat{x}_p, 1, \dots, 1 \\ y_1, \dots \quad , y_{p+s} \end{pmatrix}.$$

Since $G_\epsilon(x, y)$ is ETP, the first determinant under the integral sign is constantly strictly positive. By virtue of Karlin, [3, Chapter 10, Theorem 2.2], the second determinant is always nonnegative and is strictly iff:

(a) When $0 \leqslant q \leqslant m - 1$,

$$y_{\nu-q} < \hat{\xi}_\nu < y_{m-q+\nu}, \qquad \nu = 1, 2, \dots, q + p + s - m,$$

(b) When $q = m$ (no x^i terms appear in the determinant)

$$y_{\nu-m} < \hat{\xi}_\nu < y_\nu, \qquad \nu = 1, 2, \dots, p + s$$

(Karlin, [3, Chapter 10, Theorem 1.1]).

Clearly, either of these sets in the y variables has positive $(p + s)$-dimensional Lebesgue measure. Therefore, the determinant on the left in (2.1) is strictly positive.

We now appeal to Theorem 3.3 in Chapter 2 of Karlin [3] in order to confirm that $F_\epsilon(x, w)$ is strictly totally positive.

If x and y are any real numbers, $G_\epsilon(x, y)$ approaches the delta function $\delta(x - y)$ as ϵ decreases to zero. Therefore,

$$\lim_{\epsilon \to 0} F_\epsilon(x, w) = K(x, w), \qquad 0 < x < 1, \quad w \in W.$$

An easy calculation, using integration by parts, shows that this relation also holds for $F_\epsilon(z, w)$, where z is an integer. For example, if $0 < w < 1$,

$$\begin{aligned} F_\epsilon(\nu, w) &= \int_0^\infty D_u{}^\nu G_\epsilon(u, y)\,|_{u=1}\, K(y, w)\, dy \\ &= \int_0^\infty (-1)^\nu D_y{}^\nu G_\epsilon(1, y)\, K(y, w)\, dy \\ &= \int_0^\infty G_\epsilon(1, y)\, D_y{}^\nu K(y, w)\, dy + \text{boundary terms.} \end{aligned}$$

The exponential decay of $G_\epsilon(x, y)$ at $y = \infty$ causes the boundary terms at infinity to vanish. Moreover, elementary properties of the Gaussian kernel imply that the boundary term at $y = 0$ involving factors of the form $D_y{}^\mu G_\epsilon(1, y)|_{y=0}$ tends to zero as ϵ approaches zero. Similar reasoning covers the case where w is an integer. Therefore

$$\lim_{\epsilon \to 0} F_\epsilon(z, w) = K(z, w), \qquad z \in Z, \quad w \in W.$$

Since $F_\epsilon(z, w)$ is STP on $Z \times W$, $K(z, w)$ is totally positive on $Z \times W$ and the proof of the lemma is complete.

Our next task will be to establish the sufficiency of conditions (1.3) and (1.4). This is done by examining four cases of different values of λ, ρ, σ, and τ (Lemmas 2–6). The necessity of (1.3) and (1.4) will be proved last.

LEMMA 2. *If* $j_\nu \leqslant i_\nu$, $\nu = 1, 2, \ldots, p$, *then*

$$K\begin{pmatrix} j_1, j_2, \ldots, j_p \\ i_1, i_2, \ldots, i_p \end{pmatrix} > 0.$$

Proof. Sufficiency is trivial if $p = 1$. Assume the validity of Lemma 2 whenever $j_p \leqslant k$; we will prove it to be correct if $j_p = k + 1$.

Observe that, if $p = k + 2$, then

$$K\begin{pmatrix} 0, & 1, \ldots, k+1 \\ l, l+1, \ldots, l+k+1 \end{pmatrix} > 0$$

when $l \geqslant 0$, $l + k + 1 \leqslant m - 1$. This fact obtains since $\{x^i\}_0^{m-1}$ forms an ETP system (Karlin, [3, Chapter 6, Corollary 1.2]). Applying Theorem 3.3, Chapter 2 of [3], as in the proof of part (i), the assertion is established for $p = k + 2$.

Therefore, to advance the induction step, it is sufficient to prove the result for $j_p = k + 1$ and $p = p' \leqslant k + 1$, assuming it to be true if $j_p \leqslant k$, or if $j_p = k + 1$ and $p \geqslant p' + 1$. (Note we are advancing a second backward induction on the size of the determinant p.) We can also assume that $i_p > k+1$ for if $i_p = k + 1$, the determinant expanded by the last row yields

$$K\binom{j_1, j_2, \ldots, j_{p-1}, k+1}{i_1, i_2, \ldots, i_{p-1}, k+1} = \eta_{k+1} K\binom{j_1, j_2, \ldots, j_{p-1}}{i_1, i_2, \ldots, i_{p-1}},$$

where $\eta_{k+1} > 0$ and the last determinant is positive by the induction hypothesis since $j_{p-1} \leqslant k$.

Let i' be the largest nonnegative integer smaller than $i_{p'}$ and not included in $\{i_\nu\}_1^{p'}$, and let j' be the smallest nonnegative integer not exceeding $k + 1$ and not included in the set $\{j_\mu\}_1^{p'}$. Such integers exist, for by assumption, $p' \leqslant k + 1$ and $i_p > k + 1$. Insert i' and j' into the sets $\{i_\nu\}_1^{p'}$ and $\{j_\mu\}_1^{p'}$, respectively, arranged in natural order. It is easy to check that the inequality $j_\nu \leqslant i_\nu$ is satisfied for the expanded sets.

According to Sylvester's determinant identity,

$$\begin{aligned} &K\binom{j_1, \ldots, j', \ldots, j_{p'-1}, k+1}{i_1, \ldots, i', \ldots, i_{p'-1}, i_{p'}} K\binom{j_1, \ldots, j_{p'-1}}{i_1, \ldots, i_{p'-1}} \\ &\quad = K\binom{j_1, \ldots, j', \ldots, j_{p'-1}}{i_1, \ldots, i', \ldots, i_{p'-1}} K\binom{j_1, \ldots, j_{p'-1}, k+1}{i_1, \ldots, i_{p'-1}, i_{p'}} - K_1 K_2 \qquad (2.2) \end{aligned}$$

where each K_1 and K_2 is nonnegative, by part (i) of Theorem 1. Since $j_{p'-1} \leqslant k$, the induction hypothesis implies that the second determinant on the left of (2.2) is strictly positive. The backward induction on p assures that the first determinant on the left of (2.2) is strictly positive.

Therefore, if

$$K\binom{j_1, \ldots, j_{p'}}{i_1, \ldots, i_{p'}} = 0,$$

we infer from the above analysis of (2.2) that $-K_1K_2 > 0$. As pointed out previously, this is impossible. Therefore, the claim in Lemma 2 is validated.

LEMMA 3. *If* $j_\mu \leqslant i_\mu$, $\mu = 1, 2, \ldots, q$, *then*

$$K\binom{j_1, \ldots, j_q, j_{q+1}, \ldots, j_{q+r}}{i_1, \ldots, i_q, \xi_1, \ldots, \xi_r} > 0.$$

Proof. The proof proceeds by induction on r, with a second, backward induction on q. For $r = 0$, Lemma 2 applies. If $q = m - r$, the largest possible value, the determinant becomes

$$K\binom{0, 1, \ldots, m-1}{i_1, \ldots, i_q, \xi_1, \ldots, \xi_r},$$

and Theorem 2.2, Chapter 10† of Karlin, [3], affirms that this determinant is strictly positive whenever $\xi_\nu < 1$, $\nu = 1,..., r$. To advance the induction, it is sufficient to prove the result true when $r = r'$ and $q = q' \leqslant m - r' - 1$, assuming it to be true whenever $r \leqslant r' - 1$, and when $r = r'$ and $q \geqslant q' + 1$.

Let i' be the largest nonnegative integer less than or equal to $m - 1$ and not contained in $\{i_\nu\}_1^{q'}$, and j', the smallest such integer not contained in $\{j_\mu\}_1^{q'+r'}$. Since $q' + r' \leqslant m - 1$, such integers exist. If i' is inserted into $\{i_\nu\}$ and j' into $\{j_\mu\}$ in natural order, the enlarged sets continue to satisfy the requisite conditions of (1.3). According to Sylvester's determinant identity,

$$
\begin{aligned}
&K\begin{pmatrix} j_1,..., j',..., j_{q'+r'} \\ i_1,..., i',..., i_{q'}, \xi_1,..., \xi_{r'} \end{pmatrix} \\
&\qquad \times K\begin{pmatrix} j_1, j_2,..., j_{q'+r'-1} \\ i_1,..., i_{q'}, \xi_1,..., \xi_{r'-1} \end{pmatrix} \\
&\quad = K\begin{pmatrix} j_1,..., j',..., j_{q'+r'-1} \\ i_1,..., i',..., i_{q'}, \xi_1,..., \xi_{r'-1} \end{pmatrix} K\begin{pmatrix} j_1,..., j_{q'+r'} \\ i_1,..., i_{q'}, \xi_1,..., \xi_{r'} \end{pmatrix} - K_1K_2
\end{aligned} \tag{2.3}
$$

where K_1 and K_2 are nonnegative determinants, as in Lemma 2. The backward induction on q and forward induction on r imply that the first and second determinants, respectively, on the left in (2.3) are strictly positive. Therefore, if

$$K\begin{pmatrix} j_1, j_2,..., j_{q'+r'} \\ i_1,..., i_{q'}, \xi_1,..., \xi_{r'} \end{pmatrix} = 0,$$

then $-K_1K_2 > 0$, an absurdity. This completes the proof of Lemma 3.

LEMMA 4. *If*

$$j_\nu \leqslant i_{p+\nu}, \qquad \nu = 1, 2,..., s, \tag{2.4}$$

$$x_\mu < \xi_{\mu+m-s-p}, \qquad \mu = 1, 2,..., p, \tag{2.5}$$

then

$$K\begin{pmatrix} x_1,..., x_p, j_1,..., j_s,..., j_{s+r} \\ i_1,..., i_p,..., i_{p+s}, \xi_1,..., \xi_r \end{pmatrix} > 0.$$

Proof. The proof of the sufficiency of the conditions is by induction on p, with a backward induction on $s + r$. The statement is true for $p = 0$, for then we are in the case of Lemma 3. If $s + r = m$, Theorem 2.2, Chapter 10 of Karlin, [3], asserts that the determinant is positive if (2.5) is satisfied; in this case, $j_\nu = \nu - 1$, so (2.4) manifestly holds. Hence, the result is confirmed when $r + s = m$.

† Theorem 2.2 is stated for the case of distinct $x_1, x_2,..., x_\lambda$ but the proof works as well for coincident x's.

To advance the induction step, we need to show that if the result is true for $p \leqslant p' - 1$, and for $p = p'$ when $r + s = t + 1$, then it is true for $p = p'$ and $r + s = t$.

Choose j' to be the smallest nonnegative integer less than m not included in $\{j_\nu\}_1^t$; then if j' is inserted into $\{j_\nu\}_1^t$, the expanded set in natural order and $\{i_\mu\}_1^{p+s}$ will still satisfy (2.4). Choose $\xi_{r+1} > x_p$ so that $\xi_r < \xi_{r+1} < 1$. Then (2.5) is valid for the expanded set of ξ's.

According to Sylvester's determinant identity,

$$K\begin{pmatrix} x_1,\dots, x_{p'}, j_1,\dots, j',\dots, j_t \\ i_1,\dots, i_{p'+s}, \xi_1,\dots, \xi_r, \xi_{r+1}\end{pmatrix} K\begin{pmatrix} x_1,\dots, x_{p'-1}, j_1,\dots, j_t \\ i_1,\dots, i_{p'+s-1}, \xi_1,\dots, \xi_r\end{pmatrix}$$
$$= K\begin{pmatrix} x_1,\dots, x_{p'}, j_1,\dots, j_t \\ i_1,\dots, i_{p'+s}, \xi_1,\dots, \xi_r\end{pmatrix} K\begin{pmatrix} x_1,\dots, x_{p'-1}, j_1,\dots, j',\dots, j_t \\ i_1,\dots, i_{p'+s-1}, \xi_1,\dots, \xi_{r+1}\end{pmatrix} - K_1K_2 \tag{2.6}$$

where K_1 and K_2 are nonnegative determinants. The determinants on the left in (2.6) are inferred to be strictly positive by invoking appropriately the induction hypothesis. Therefore, if the first determinant on the right were zero, we would have a contradiction.

LEMMA 5. *If*

$$\xi_\nu < x_{q+\nu}, \qquad \nu = 1, 2,\dots, r, \tag{2.7}$$

$$x_\mu < \xi_{m+\mu-q}, \qquad \mu = 1, 2,\dots, q + r, \tag{2.8}$$

then

$$K\begin{pmatrix} x_1,\dots, x_q, x_{q+1},\dots, x_{q+r}, j_1,\dots, j_p \\ i_1,\dots, i_q, \xi_1,\dots, \xi_r, \xi_{r+1},\dots, \xi_{r+p}\end{pmatrix} > 0.$$

Proof. The proof proceeds by induction on r, with a second, backward induction on p. If $r = 0$, we are in the situation of Lemma 4; the desired result is achieved. If $p = m$, apply Theorem 2.2, Chapter 10 of Karlin [3], as in the proof of Lemma 3, to conclude that the determinant is strictly positive provided (2.7) and (2.8) prevail. If $p = 0$ Theorem 2.2 also applies.

Assume the validity of the result when $r \leqslant r' - 1$, or if $r = r'$ and $p \geqslant p' + 1$. Let j' be any nonnegative integer, at most $m - 1$, not in $\{j_\nu\}_1^{p'}$, and let $\xi_{r'+p'+1}$ be chosen so that $\xi_{r'+p'} < \xi_{r'+p'+1} < 1$. Then conditions (2.7) and (2.8) persist for $\{x_\nu\}_1^{q+r'}$ and $\{\xi_\mu\}_1^{r'+p'+1}$.

Sylvester's determinant identity yields

$$K\begin{pmatrix} x_1,\dots, x_{q+r'}, j_1,\dots, j',\dots, j_{p'} \\ i_1,\dots, i_q, \xi_1,\dots, \xi_{r'+p'+1}\end{pmatrix} K\begin{pmatrix} x_1,\dots, x_{q+r'-1}, j_1,\dots, j_{p'} \\ i_1,\dots, i_q, \xi_1,\dots, \xi_{r'+p'-1}\end{pmatrix}$$
$$= K\begin{pmatrix} x_1,\dots, x_{q+r'}, j_1,\dots, j_{p'} \\ i_1,\dots, i_q, \xi_1,\dots, \xi_{r'+p'}\end{pmatrix} K\begin{pmatrix} x_1,\dots, x_{q+r'-1}, j_1,\dots, j',\dots, j_{p'} \\ i_1,\dots, i_q, \xi_1,\dots, \xi_{r'+p'-1}, \xi_{r'+p'+1}\end{pmatrix} - K_1K_2 \tag{2.9}$$

where K_1 and K_2 are nonnegative determinants, by part (i) of the theorem. The determinants on the left in (2.9) are both strictly positive, by the induction hypothesis. Therefore, we infer as before that the first determinant on the right cannot be zero.

This completes the proof that (1.3) and (1.4) entail that the determinant in (1.2) is positive.

Proof that conditions (1.3) *and* (1.4) *are necessary for strict inequality in* (1.2). Suppose, contrary to the second condition of (1.3), that

$$x_t \geqslant \xi_{m-\sigma+t} \qquad \text{for some } t,\ 1 \leqslant t \leqslant \lambda.$$

Then

$$x_\mu \geqslant \xi_\nu , \qquad \mu = t, t+1,..., \lambda;\ \nu = 1, 2,..., m-\sigma+t.$$

We must have $m - \sigma + t \geqslant 1$, for otherwise, $\sigma \geqslant m + t$, but $\sigma \leqslant m$, by hypothesis. Note that $u_{i_\alpha}(x)$, $\alpha = 1, 2,..., \sigma$, is a solution of $y^{(m)} = 0$ on $(x_t, 1]$ (recall that $u_i(x) = x^i$), as is $K(x, \xi_\nu)$, $\nu = 1, 2,..., m - \sigma$, since $\xi_\nu < x_t$ for these values of ν. Furthermore, these m functions are linearly independent, for according to the sufficiency of conditions (1.3) and (1.4),

$$K\begin{pmatrix}\eta_1 ,..., & & & & & \eta_m \\ i_1, i_2 ,..., i_\sigma, \xi_1, \xi_2 ,..., \xi_{m-\sigma}\end{pmatrix} > 0$$

whenever $x_t \leqslant \eta_1 < \cdots < \eta_m < 1$. Therefore, each of the functions $K(x, \xi_\nu)$, $\nu = m - \sigma + 1,..., m - \sigma + t$, which is also a solution of $y^{(m)} = 0$ for $x \geqslant x_t$, can be represented as a linear combination of the functions $u_{i_\alpha}(x)$, $\alpha = 1, 2,..., \sigma$, and $K(x, \xi_\nu)$, $\nu = 1, 2,..., m - \sigma$. The same representation applies for $D_x^{\,j} K(x, \xi_\nu)\,|_{x=1-}$, $j = 0, 1,..., m-1$, in terms of the corresponding derivatives of these functions. It follows in this case, that there are linear combinations of the first m columns of

$$K\begin{pmatrix}x_1 ,..., x_\lambda, j_1 ,..., j_p \\ i_1 ,..., i_\sigma, \xi_1 ,..., \xi_\tau\end{pmatrix}$$

which, added to the appropriate columns, will annihilate all elements in columns $m+1$, $m+2,..., m+t$ except, possibly, those in the first $t-1$ rows. The resulting determinant has the form

$$\begin{array}{r|c|c|c|}
 & \xi_{m-\sigma} & \xi_{m-\sigma+t+1} & \\
\hline
x_1 & & & \\
x_{t-1} & & & \\
\hline
 & & \begin{matrix}0 & \cdots & 0\\ \vdots & & \vdots \\ 0 & \cdots & 0\end{matrix} & \\
\end{array}$$

By a standard argument, we deduce that the t columns corresponding to $\xi_{m-\sigma+1}, \ldots, \xi_{m-\sigma+t}$ are linearly dependent, so the determinant is zero.

Suppose next that $\sigma > \lambda$ and $j_t > i_{\lambda+t}$ for some t, $1 \leqslant t \leqslant \sigma - \lambda$. Then $j_\nu > i_\mu$, $\nu = t, t+1, \ldots, \sigma - \lambda$; $\mu = 1, 2, \ldots, \lambda + t$. We have

$$K(j_\nu, i_\mu) = (D^{j_\nu} u_{i_\mu})(x) \,|_{x=1} = 0, \ \nu = t, t+1, \ldots, \sigma - \lambda; \ \ \mu = 1, 2, \ldots, \lambda + t.$$

Then the first $\lambda + t$ columns of the determinant have nonzero elements only in the first $\lambda + t - 1$ rows, and hence are linearly dependent.

To establish the necessity of condition (1.4) it remains to examine the possibility that $\sigma < \lambda$ and $x_{\sigma+t} \leqslant \xi_t$ for some t, $1 \leqslant t \leqslant \lambda - \sigma$. Then

$$x_\mu \leqslant \xi_\nu, \ \mu = 1, 2, \ldots, \sigma + t; \ \nu = t, t+1, \ldots, \tau,$$

so

$$K(x_\mu, \xi_\nu) = 0, \ \mu = 1, 2, \ldots, \sigma + t; \ \nu = t, t+1, \ldots, \tau.$$

The first $\sigma + t$ rows of the determinant have nonzero elements only in the first $\sigma + t - 1$ columns, and hence are linearly dependent.

This completes the proof of Theorem 1.

3. Interpolation by Splines Subject to Boundary Conditions

Theorem 1 can be used to solve the following problem: Interpolate data at the points $X = \{x_\nu, \nu = 1, 2, \ldots, \lambda, 0 < x_\nu < 1\}$ by polynomials splines (see (0.1)) of degree $m - 1$ with prescribed knots $\{\xi_\nu\}_1^r$,

$$0 < \xi_1 < \xi_2 < \cdots < \xi_r < 1,$$

satisfying the boundary conditions

$$\beta_0 : \sum_{\mu=0}^{m-1} A_{\nu\mu} \varphi^{(\mu)}(0) = 0, \qquad \nu = 1, 2, \ldots, p, \tag{3.1}$$

$$\beta_1 : \sum_{\mu=0}^{m-1} B_{\nu\mu} \varphi^{(\mu)}(1) = 0, \qquad \nu = 1, 2, \ldots, q, \tag{3.2}$$

In other words, we wish to determine coefficients $\{a_i\}_0^{m-1}$ and $\{c_i\}_1^r$ so that

$$S(x) = \sum_{i=0}^{m-1} a_i x^i + \sum_{i=1}^{r} c_i (x - \xi_i)_+^{m-1} \tag{3.3}$$

satisfies the boundary conditions $\beta_0 \cap \beta_1$, and

$$S(x_\nu) = y_\nu, \qquad \nu = 1, 2,..., \lambda,$$

where $\{y_\nu\}_1^\lambda$ is given. The problem clearly involves solving a system of linear equations. To guarantee the possibility of unique interpolation for arbitrarily prescribed data, it is manifestly required that

$$p + q + \lambda = m + r.$$

Assume that the matrices $\| A_{\nu\mu} \|$, $\| B_{\nu\mu} \|$ *in* (3.1) *and* (3.2) *satisfy postulate* I

THEOREM 2. *Let the knots* $\{\xi_\nu\}_1^r$ *and the points* $X = \{x_\nu\}_1^\lambda$ *be given, satisfying* $0 < \xi_1 < \xi_2 < \cdots < \xi_r < 1$ *and* $0 < x_1 < x_2 < \cdots < x_\lambda < 1$ *where* $p + q + \lambda = r + m$. *Unique interpolation at* X *occurs by a spline* $S(x)$ *of class* $\mathscr{S}_{m,r}(\beta_0 \cap \beta_1)$ *provided postulate I holds and then if and only if* $0 \leqslant i_1 < i_2 < \cdots < i_p \leqslant m - 1$ *and* $0 \leqslant j_1 < j_2 < \cdots < j_q \leqslant m - 1$ *exist satisfying*

$$A\begin{pmatrix} 1, 2,..., p \\ i_1, i_2,..., i_p \end{pmatrix} \neq 0 \qquad \text{and} \qquad B\begin{pmatrix} 1, 2,..., q \\ j_1, j_2,..., j_q \end{pmatrix} \neq 0 \tag{3.4}$$

while $\{x_\nu\}_1^\lambda$, $\{j_\nu\}_1^q$, $\{i_\nu'\}_1^{m-p}$, $\{\xi_\nu\}_1^r$ *obey the restrictions of Theorem* 1. (*Here* $\{i_\nu'\}_1^{m-p}$ *denotes the complementary set of indices to* $\{i_\nu\}_1^p$ *among the collection* $\{0, 1, 2,..., m - 1\}$.)

It is worh exhibiting some important examples of boundary conditions $\beta_0 \cap \beta_1$ fulfilling the conditions of Postulate I and the cases of validity of Theorem 2 for them. Of frequent interest is the situation of a "full set" of boundary conditions, i.e., where $p + q = m$.

EXAMPLE A. $q = m - p$. If $\lambda = q$, then the stipulation $r + m = \lambda + p + q$ entails $r = \lambda = q$. If the boundary conditions obey Postulate I then unique interpolation is possible for any sets of prescribed points x's and knots ξ's satisfying (1.3), which in the case at hand reduces to $x_\mu < \xi_{\mu+p}$, $\mu = 1, 2,..., q$.

EXAMPLE B. $q = m - p$, $\lambda > q$. Assuming Postulate I holds, unique interpolation is possible provided only that $\{\xi_\mu\}_1^r$ and $\{x_\mu\}_1^\lambda$ are specified to satisfy

$$x_\mu < \xi_{\mu+p}, \qquad \mu = 1, 2,..., \lambda - p,$$
$$\xi_\mu < x_{q+\mu}, \qquad \mu = 1, 2,..., \lambda - q.$$

EXAMPLE C. $q = m - p$, $\lambda < q$. Unique interpolation holds for all choices of the x's and ξ's provided there exist

$$0 \leqslant i_1 < i_2 < \cdots < i_p \leqslant m - 1,$$
$$0 \leqslant j_1 < j_2 < \cdots < j_{m-p} \leqslant m - 1,$$

for which

$$A\begin{pmatrix} 1,..., p \\ i_1 ,..., i_p \end{pmatrix} \neq 0, \qquad B\begin{pmatrix} 1,..., m-p \\ j_1 ,..., j_{m-p} \end{pmatrix} \neq 0,$$

and

$$j_\mu \leqslant i'_{\lambda+\mu}, \qquad \mu = 1, 2,..., q - \lambda,$$
$$x_\mu \leqslant \xi_{\mu+p} \qquad \mu = 1,..., \lambda - p$$

where $\{i_1',..., i'_{m-p}\}$ is the set of complementary indices to $\{i_1 ,..., i_p\}$ in the set $\{0, 1,..., m - 1\}$.

The next example embraces a further specialization of wide interest.

EXAMPLE D. Let $m = 2n$, $p = q = n$. Let $\beta_0 \cap \beta_1$ correspond to the simple boundary conditions $S^{(i_1)}(0) = S^{(i_2)}(0) = \cdots = S^{(i_n)}(0) = 0$ and $S^{(j_1)}(1) = S^{(j_2)}(1) = \cdots = S^{(j_n)}(1) = 0$ where $0 \leqslant i_1 < i_2 < \cdots < i_n \leqslant 2n - 1$ and $0 \leqslant j_1 < j_2 < \cdots < j_n \leqslant 2n - 1$. Suppose $x_t = \xi_t$, $t = 1,..., \lambda$ $(\lambda = r)$. Let $\{i_1',..., i_n'\}$ denote the complementary indices to $\{i_1 ,..., i_n\}$ in $\{0,1,... 2n - 1\}$. According to Theorem 2, unique interpolation holds if and only if

$$j_\mu \leqslant i'_{\lambda+\mu}, \qquad \mu = 1, 2,..., n - r. \tag{3.5}$$

In particular, if $j_1 = i_1 = 0, j_2 = i_2 = 1,..., j_n = i_n = n - 1$, then unique interpolation at the knots holds. On the other hand, if $j_1 = i_1 = n, j_2 = i_2 = n + 1,..., j_n = i_n = 2n - 1$, then inspection of (3.5) reveals that unique interpolation at the knots is possible if and only if $r \geqslant n$ (i.e., the presence of at least n knots).

Proof of Theorem 2. *Let* $u_i(x) = x^i$, $\varphi_m(x; \xi) = (x - \xi)_+^{m-1}$. The boundary conditions and interpolation for zero data lead to the following set of linear equations:

$$\sum_{i=0}^{m-1} a_i \sum_{\nu=0}^{m-1} A_{\mu\nu} D^\nu u_i(0) = 0, \qquad \mu = 1, 2,..., p, \tag{3.6}$$

$$\sum_{i=0}^{m-1} a_i u_i(x_\alpha) + \sum_{i=1}^{r} c_i \varphi_m(x_\alpha ; \xi_i) = 0, \qquad \alpha = 1, 2,..., \lambda, \tag{3.7}$$

$$\sum_{i=0}^{m-1} a_i \sum_{\nu=0}^{m-1} B_{\mu\nu} D^\nu u_i(1) + \sum_{i=1}^{r} c_i \sum_{\nu=0}^{m-1} B_{\mu\nu} D^\nu \varphi_m(x; \xi_i) \big|_{x=1} = 0, \tag{3.8}$$
$$\mu = 1, 2,..., q.$$

In (3.6), use the fact that $D^\nu u_i(0) = \delta_{\nu i} i!$ (Kronecker delta); then the matrix of the equations can be written as

$$\left\| \begin{matrix} A_{10}0! & \cdots & A_{1,m-1}(m-1)! & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{p,0}0! & \cdots & A_{p,m-1}(m-1)! & 0 & \cdots & 0 \\ u_0(x_1) & \cdots & u_{m-1}(x_1) & \varphi_m(x_1\,;\,\xi_1) & \cdots & \varphi_m(x_1\,;\,\xi_r) \\ \vdots & & \vdots & \vdots & & \vdots \\ u_0(x_\lambda) & \cdots & u_{m-1}(x_\lambda) & \varphi_m(x_\lambda\,;\,\xi_1) & \cdots & \varphi_m(x_\lambda\,;\,\xi_r) \\ \sum_{\nu=0}^{m-1} B_{1\nu}D^\nu u_0(1) & \cdots & \sum_{\nu=0}^{m-1} B_{1\nu}D^\nu u_{m-1}(1) & \sum_{\nu=0}^{m-1} B_{1\nu}D_x^\nu \varphi_m(1;\xi_1) & \cdots & \sum_{\nu=0}^{m-1} B_{1\nu}D_x^\nu \varphi_m(1;\xi_r) \\ \vdots & & \vdots & \vdots & & \vdots \\ \sum_{\nu=0}^{m-1} B_{q\nu}D^\nu u_0(1) & \cdots & \sum_{\nu=0}^{m-1} B_{q\nu}D^\nu u_{m-1}(1) & \sum_{\nu=0}^{m-1} B_{q\nu}D_x^\nu \varphi_m(1;\xi_1) & \cdots & \sum_{\nu=0}^{m-1} B_{q\nu}D_x^\nu \varphi_m(1;\xi_r) \end{matrix} \right\|.$$

To find the determinant of this matrix, we invoke the Laplace expansion by minors on the first p rows. (It is convenient to employ the notation $\tilde{A} = \| A_{\mu\nu}(-1)^\nu \|$.) Thereby the determinant is expressed as

$$(-1)^{p(p+1)/2} \sum_{0\leqslant i_1<\cdots<i_p\leqslant m-1} \tilde{A}\begin{pmatrix} 1,\dots, p \\ i_1,\dots, i_p \end{pmatrix} \left(\prod_{j=1}^{p} i_j\,! \right) D \begin{pmatrix} p+1,\dots, p+q+\lambda \\ i_1',\dots, i_{m-p}',\ \xi_1,\dots,\xi_r \end{pmatrix} \tag{3.9}$$

where the second determinant in the sum is that drawn from the last $q+\lambda$ rows, and columns $i_1',\dots, i_{m-p}'$, $m+1,\dots, m+r$ of the original matrix (recall that $\{i_\nu'\}_1^{m-p}$ is the complementary set to $\{i_\nu\}_1^p$ from $\{0, 1,\dots, m-1\}$). Note that the last $\lambda+q$ rows of the original matrix can be written

$$\left\| \begin{matrix} I_\lambda & 0_{\lambda m} \\ 0_{q\lambda} & \| B_{\mu\nu} \|_{\mu=1,\nu=0}^{q,m-1} \end{matrix} \right\| \left\| \begin{matrix} \| u_j(x_i) \|_{i=1,j=0}^{\lambda,m-1} & \| \varphi_m(x_i\,;\,\xi_j) \|_{i=1,j=1}^{\lambda,r} \\ \| D^\nu u_j(1) \|_0^{m-1} & \| D^\nu \varphi_m(1;\xi_j) \|_{\nu=0,j=1}^{m-1,r} \end{matrix} \right\|.$$

Therefore, by the Cauchy–Binet formula (see, for example Karlin, Ref. [3, Chapter 0]), we have

$$D \begin{pmatrix} p+1,\dots, p+q+\lambda \\ i_1',\dots, i_{m-p}',\ \xi_1,\dots,\xi_r \end{pmatrix} = \sum_{0\leqslant j_1<\cdots<j_q\leqslant m-1} B \begin{pmatrix} 1,\dots, q \\ j_1,\dots, j_q \end{pmatrix}$$

$$\times K \begin{pmatrix} x_1,\dots, x_\lambda, & j_1,\dots, j_q \\ i_1',\dots, i_{m-p}', & \xi_1,\dots,\xi_r \end{pmatrix}. \tag{3.10}$$

According to Theorem 1, $K(z, w)$ is totally positive, and hence the last determinant in (3.10) is always nonnegative. By the hypothesis of Theorem 2, $\tilde{A}$ is sign-consistent of order p, and B is sign-consistent of order q. Therefore, all terms in the sum in (3.10) and (3.9) have the same sign, or are zero. By Theorem 1, the conditions in the hypothesis of Theorem 2 are precisely the conditions that some term in (3.9) is nonzero. Hence, these are precisely the conditions under which the matrix of coefficients of the equations (3.6)–(3.8) has a nonzero determinant. This completes the proof of Theorem 2.

4. Total Positivity Properties of Green's Functions of Differential Operators with Boundary Conditions

Consider a differential operator, acting on functions v of continuity class $C^m[0, 1]$, of the form (0.3), viz.,

$$L_m v = D_m D_{m-1} \cdots D_1 v, \quad D_i v = \frac{d}{dx} \frac{1}{w_i(x)} v, \qquad i = 1, 2, \ldots, m, \quad (4.1)$$

coupled with the boundary conditions (cf. (0.8))

$$\begin{aligned} \beta_0 &: \sum_{\mu=0}^{m-1} A_{\nu\mu}(D^\mu v)(0) = 0, \qquad \mu = 1, \ldots, p, \\ \beta_1 &: \sum_{\mu=0}^{m-1} B_{\lambda\mu}(D^\mu v)(1) = 0, \qquad \lambda = 1, \ldots, m-p, \end{aligned} \quad (4.2)$$

where, in this general setting, $D^0 = I$, $D^\mu = D_\mu D_{\mu-1} \cdots D_1$, $\mu = 1, 2, \ldots, m-1$.

The functions $w_i(x)$ are prescribed as in (0.3). A natural basic set of solutions of $L_m y = 0$ comprise $\{u_i(x)\}_{i=0}^{m-1}$ defined in (0.4) and the fundamental solution is $\phi(x; \xi) = \phi_m(x; \xi)$, whose explicit expression is exhibited in (0.5).

The boundary conditions $\beta_0 \cap \beta_1$ are assumed to be of the type fulfilling Postulate I. Thus

(a) $\tilde{A} = \| A_{\nu\mu}(-1)^\mu \|$ is sign-consistent of order p (SC_p) and of rank p;

(b) $B = \| B_{\lambda\mu} \|$ is SC_q and of rank q.

We further assume that the only solution of $L_m v = 0$ satisfying $\beta = \beta_0 \cap \beta_1$ is the trivial solution (see Theorem 3 below). This fact guarantees the existence of a Green's function $G(x, \xi)$ with the properties that $L_m v = f$, for f continuous a suitable f, can be uniquely solved with $v \in \beta$, the solution admitting the integral representation

$$v(x) = \int_0^1 G(x, \xi) f(\xi)\, d\xi.$$

The Green's function $G(x, \xi)$ displays the following characteristic properties:

(i) $G(x, \xi)$ is C^{m-2} on the unit square $0 \leqslant x, \xi \leqslant 1$ and C^m in the regions $0 \leqslant x < \xi \leqslant 1$ and $0 \leqslant \xi < x \leqslant 1$.

(ii) $L_m^{(x)} G(x, \xi) = 0$ $(0 < x < \xi)$ $L_m^{(x)} G(x, \xi) = 0$ $(\xi < x < 1)$.

(iii) For each ξ, $G(x, \xi)$ satisfies $\beta = \beta_0 \cap \beta_1$.

(iv) G displays the characteristic jump discontinuity in its $(m-1)$-th derivative. Specifically,

$$\frac{1}{w_m(x)} [D_x^{(m-1)} G(x; x-) - D_x^{(m-1} G(x; x+)] = 1.$$

Karon [7], adapting a method of the author [3, Chapter 10, Section 6], established that $G(x, \xi)$ is a sign-regular kernel and ascertained conditions for the nonvanishing of the determinants

$$G^* \binom{x_1, x_2, \ldots, x_k}{\xi_1, \xi_2, \ldots, \xi_k}. \tag{4.3}$$

We shall obtain this result as a by-product of Theorem 1; in fact, we can determine the actual sign of (4.3), which is not accessible by the previous methods.

Proceeding to this task, let $A^{(1)}, A^{(2)}, \ldots, A^{(p)}, B^{(1)}, B^{(2)}, \ldots, B^{(q)}$, $q = m - p$, denote the row vectors of the matrices $A = \| A_{\nu\mu} \|$ and $B = \| B_{\lambda\mu} \|$, respectively. Introduce the vectors $\{\bar{u}^{(i)}\}$ defined in component form to be

$$\bar{u}^{(i)} = (u_i(0), D^1 u_i(0), D^2 u_i(0), \ldots, D^{(m-1)} u_i(0)), \quad i = 0, 1, \ldots, m-1,$$

and similarly define

$$\tilde{u}^{(i)} = (u_i(1), D^1 u_i(1), D^2 u_i(1), \ldots, D^{m-1} u_i(1)), \qquad i = 0, 1, \ldots, m-1.$$

Also, let for $0 < \xi < 1$

$$\bar{\varphi}(\xi) = (\phi(0; \xi), D_x{}^1 \phi(0; \xi), \ldots, D_x^{m-1} \phi(0; \xi)) = \mathbf{0} \quad \text{(the zero vector)},$$

$$\tilde{\varphi}(\xi) = (\phi(1; \xi), D_x{}^1 \phi(1; \xi), \ldots, D_x^{m-1} \phi(1; \xi)).$$

(Here $D_x^{(i)}$ signifies that the differential operation $D^{(i)}$ is performed with respect to the variable x.) Let $(A^k, \bar{u}^{(k)})$ denote the inner product of the indicated vectors. Finally, let Δ denote the cofactor of $\phi(x; \xi)$ in the determinant on the right of (4.4) below.

An explicit representation of Green's function in terms of $\{u_i(x)\}_{i=0}^{m-1}$ and $\phi(x;\xi)$ is readily verified to be

$$G(x,\xi)=\frac{1}{w_m(\xi)\Delta}\begin{vmatrix} (A^{(1)},\bar{u}^{(0)}) & (A^{(1)},\bar{u}^{(1)}) & \cdots & (A^{(1)},\bar{u}^{(m-1)}) & (A^{(1)},\bar{\varphi}(\xi)) \\ (A^{(2)},\bar{u}^{(0)}) & (A^{(2)},\bar{u}^{(1)}) & \cdots & (A^{(2)},\bar{u}^{(m-1)}) & (A^{(2)},\bar{\varphi}(\xi)) \\ \vdots & \vdots & & \vdots & \vdots \\ (A^{(p)},\bar{u}^{(0)}) & (A^{(p)},\bar{u}^{(1)}) & \cdots & (A^{(p)},\bar{u}^{(m-1)}) & (A^{(p)},\bar{\varphi}(\xi)) \\ (B^{(1)},\tilde{u}^{(0)}) & (B^{(1)},\tilde{u}^{(1)}) & \cdots & (B^{(1)},\tilde{u}^{(m-1)}) & (B^{(1)},\tilde{\varphi}(\xi)) \\ \vdots & \vdots & & \vdots & \vdots \\ (B^{(q)},\tilde{u}^{(0)}) & (B^{(q)},\tilde{u}^{(1)}) & \cdots & (B^{(q)},\tilde{u}^{(m-1)}) & (B^{(q)},\tilde{\varphi}(\xi)) \\ u_0(x) & u_1(x) & \cdots & u_{m-1}(x) & \phi(x;\xi) \end{vmatrix}. \tag{4.4}$$

An application of the Sylvester determinant identity produces the formula

$$G\begin{pmatrix} x_1, x_2, \ldots, x_r \\ \xi_1, \xi_2, \ldots, \xi_r \end{pmatrix} = \frac{1}{[\prod_{i=1}^{r} w(\xi_i)]\,\Delta} D$$

where

$$D=\begin{vmatrix} (A^{(1)},\bar{u}^{(0)}) & \cdots & (A^{(1)},\bar{u}^{(m-1)}) & (A^{(1)},\bar{\varphi}(\xi_1)) & \cdots & (A^{(1)},\bar{\varphi}(\xi_r)) \\ \vdots & & \vdots & \vdots & & \vdots \\ (A^{(p)},\bar{u}^{(0)}) & \cdots & (A^{(p)},\bar{u}^{(m-1)}) & (A^{(p)},\bar{\varphi}(\xi_1)) & \cdots & (A^{(p)},\bar{\varphi}(\xi_r)) \\ (B^{(1)},\tilde{u}^{(0)}) & \cdots & (B^{(1)},\tilde{u}^{(m-1)}) & (B^{(1)},\tilde{\varphi}(\xi_1)) & \cdots & (B^{(1)},\tilde{\varphi}(\xi_r)) \\ \vdots & & \vdots & \vdots & & \vdots \\ (B^{(q)},\tilde{u}^{(0)}) & \cdots & (B^{(q)},\tilde{u}^{(m-1)}) & (B^{(q)},\tilde{\varphi}(\xi_1)) & \cdots & (B^{(q)},\tilde{\varphi}(\xi_r)) \\ u_0(x_1) & \cdots & u_{m-1}(x_1) & \phi(x_1;\xi_1) & \cdots & \phi(x_1;\xi_r) \\ \vdots & & \vdots & \vdots & & \vdots \\ u_0(x_r) & \cdots & u_{m-1}(x_r) & \phi(x_r;\xi_1) & \cdots & \phi(x_r;\xi_r) \end{vmatrix}.$$

Expanding Δ as in the proof of Theorem 2 produces

$$\Delta = (-1)^{p(p+1)/2} \sum_{0\le i_1<i_2<\cdots<i_p\le m-1} \Big(\prod_{\nu=1}^{p} i_\nu\,!\Big)\, \tilde{A}\begin{pmatrix} 1, 2, \ldots, p \\ i_1, i_2, \ldots, i_p \end{pmatrix}$$
$$\times \sum_{j_1<j_2<\cdots<j_q} B\begin{pmatrix} 1, 2, \ldots, q \\ j_1, j_2, \ldots, j_q \end{pmatrix} K\begin{pmatrix} j_1, j_2, \ldots, j_q \\ i_1', i_2', \ldots, i_{m-p}' \end{pmatrix} \tag{4.5}$$

where $\{i_\mu'\}_1^{m-p}$ are the complementary indices to $\{i_\mu\}_1^p$. Inspection of the expression (4.5) and reference to Theorem 1 reveal the following criteria for the existence of a Green's function.

THEOREM 3. *The differential operator L_m in* (4.1) *with boundary conditions* (4.2) *subject to Postulate I possesses a Green's function iff $\Delta \neq 0$ or, equivalently, iff there exist sets of indices $\{i_\mu\}_1^p$ and $\{j_\nu\}_1^q$ $(q = m - p$, $0 \leqslant j_1 < j_2 < \cdots < j_q \leqslant m - 1)$ such that*

$$A\begin{pmatrix} 1, 2, \ldots, p \\ i_1, i_2, \ldots, i_p \end{pmatrix} \neq 0, \quad B\begin{pmatrix} 1, 2, \ldots, q \\ j_1, j_2, \ldots, j_q \end{pmatrix} \neq 0, \quad \text{and} \quad j_\mu \leqslant i_\mu' \quad (\mu = 1, 2, \ldots, q),$$

where $\{i_\mu'\}_1^q$ are the complementary indices to $\{i_\mu\}_1^p$ in $\{0, 1, \ldots, m - 1\}$.

The statement of Theorem 3 formalizes the assumption concerning the existence of a Green's function made at the start of this section.

We next evaluate the determinant D. Permuting the rows involving the vectors $\{B^{(j)}\}$ to the bottom, expanding again as in the proof of Theorem 2, and combining with (4.5), we obtain

$$\begin{aligned} G\begin{pmatrix} x_1, x_2, \ldots, x_r \\ \xi_1, \xi_2, \ldots, \xi_r \end{pmatrix} = \Bigg[(-1)^{r(m-p)} & \sum_{0 \leqslant i_1 < i_2 < \cdots < i_p \leqslant m-1} \tilde{A}\begin{pmatrix} 1, 2, \ldots, p \\ i_1, i_2, \ldots, i_p \end{pmatrix} \\ & \times \sum_{j_1 < j_2 < \cdots < j_q} B\begin{pmatrix} 1, \ldots, q \\ j_1, \ldots, j_q \end{pmatrix} K\begin{pmatrix} x_1, \ldots, x_r, j_1, \ldots, j_q \\ i_1', \ldots, i_q', \xi_1, \ldots, \xi_r \end{pmatrix} \Bigg] \\ & \div \Bigg[\Pi_{i=1}^r [w(\xi_i)] \sum_{i_1 < i_2 < \cdots < i_p} \tilde{A}\begin{pmatrix} 1, 2, \ldots, p \\ i_1, i_2, \ldots, i_p \end{pmatrix} \\ & \times \sum_{j_1 < j_2 < \cdots < j_q} B\begin{pmatrix} 1, 2, \ldots, q \\ j_1, j_2, \ldots, j_q \end{pmatrix} K\begin{pmatrix} j_1, \ldots, j_q \\ i_1', \ldots, i_{m-p}' \end{pmatrix} \Bigg]. \end{aligned} \tag{4.6}$$

The sign-regularity properties of $G(x, \xi)$ can be read off from this formula, in view of the exact delineation of the total positivity character of $K(z, w)$ in Theorem 1. We sum up in the following theorem. (For obvious reasons it is convenient to multiply the operator L_m by the constant factor $(-1)^{m-p}$.)

THEOREM 4. *Consider the differential operator $(-1)^{m-p} L_m$ with boundary conditions* (4.2) *of the type fulfilling Postulate* I. *Assume the associated Green's function $M(x, \xi)$ exists* (*or, equivalently, the conditions of Theorem* 3 *hold*). *Then*

$$M^*\begin{pmatrix} x_1, x_2, \ldots, x_r \\ \xi_1, \xi_2, \ldots, \xi_r \end{pmatrix} \geqslant 0 \qquad \text{for} \quad 0 < \begin{matrix} x_1 \leqslant x_2 \leqslant \cdots \leqslant x_r \\ \xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_r \end{matrix} \leqslant 1 \tag{4.7}$$

(*i.e., $M(x, \xi)$ is* TP) *and strict inequality holds if and only if*

$$\begin{aligned} \xi_\mu &< x_{\mu+q}, \qquad \mu = 1, 2, \ldots, r - q, \\ x_\mu &< \xi_{\mu+p}, \qquad \mu = 1, 2, \ldots, r - p. \end{aligned} \tag{4.8}$$

Some special cases of Theorem 4 with distinct x's and ξ's are stated in Krein [8].

Applications of Theorem 4 to the oscillation theory of solutions of differential operators of type (4.1) will be presented elsewhere. Consult also Karlin, Ref. [3, Chapter 10, Section 6], for further discussion on the significance and relevance of this theorem to vibrating coupled mechanical systems.

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